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Abstract

Alfvén's perturbation method for treating the motion of a point charge in a magnetic field consists of three elements: 1) The field is regarded, in zero order, as homogeneous. In the next order, the zero-order motion is replaced by its averaged effect, a magnetic dipole, and the interaction of this 'equivalent dipole' with the inhomogeneities of the field constitutes the perturbation. 2) The moment of the 'equivalent dipole' is constant. 3) If the field is axially symmetric, the 'equivalent dipole' moves on the surface generated by rotating a line of force about the polar axis.

Alfvén develops his method by the analysis of several special cases. In this report, we have re-derived his perturbation scheme from a viewpoint that emphasizes general dynamical principles. Thus: 1) is derived directly from Hamilton's principle, 2) is proved by the adiabatic theorem, after noting that the moment of the 'equivalent dipole' is an action variable, while 3) is related to the conservation of angular momentum.

Our remarks under 2) and 3) are intended as an extension of Alfvén's discussion.

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1. Introduction

The basic elementary process in magneto-hydrodynamics is the motion of a point charge in a magnetic field. For a uniform magnetic field, the solution is trivial; for a more general magnetic field, the solution can be quite complicated. An intermediate situation has been considered by Alfvén[1] namely, the case where the variation of the field may be regarded as a small perturbation of a uniform field.

In Alfvén's method, the field is regarded, in zero order, as homogeneous. In the next order, the zero-order motion is replaced by its average effect, a magnetic dipole, and the interaction of this 'equivalent dipole' with the inhomogeneities of the field constitutes the perturbation.

Alfvén presents his method by analyzing several special cases. In this report, we have re-derived his perturbation scheme from a viewpoint that emphasizes general dynamical principles.

Our discussion extends Alfvén's treatment in two respects: First, the circumstances under which the moment of 'equivalent dipole' is constant are summarized concisely by noting that it is an action variable, and therefore is unaffected by any changes that are 'adiabatically slow' compared to the zero-order motion. Second, if the field is axially symmetric, it is shown that conservation of angular momentum requires that the 'equivalent dipole' must move on the surface generated by rotating a line of a force about the polar axis.

2. The zero-order motion

We consider a point particle (charge ε, mass m) in a magnetic field, $\underline{H}$. The position vector of the particle is the sum of a slowly varying part, $\underline{R}$, and a rapidly varying part, $\underline{r}$. $\underline{r}$ is used to describe the zero-order motion and $\underline{R}$

to describe the first-order motion. In an homogeneous field, $\underline{R}$ is constant.

The zero-order motion regards the magnetic field as approximately homogeneous. The particle then makes excursions of the order $\underline{r}$ about a point $\underline{R}$. Thus, the approximation of homogeneity is justified if

$$\underline{r} \cdot \nabla_{\underline{R}}\, \underline{H}(\underline{R}) \ll \underline{H}(\underline{R}) \quad . \tag{2.1}$$

If

$$H(\underline{R}) \sim \frac{1}{R^n}$$

(2.1) requires that

$$\frac{R}{r} \gg n$$

or, since $n \sim 1$ in most cases, our perturbation theory will be applicable if

$$\frac{R}{r} \gg 1 \quad . \tag{2.2}$$

So far, we have held $\underline{R}$ constant during the motion and have considered the variation of the field as $\underline{r}$ varies. Actually $\underline{R}$ also varies if the field is not homogeneous. If the consequent magnetic field variation is to be negligible, it is sufficient to add to the condition (2.1):

$$\tau\, \dot{\underline{R}} \ll \underline{r} \tag{2.3}$$

where τ is the period associated with the zero-order motion. (2.3) implies that

$$\dot{\underline{R}} \ll \dot{\underline{r}} \tag{2.4}$$

while (2.2) and (2.3) imply that

(2.5) $$\tau \underline{\dot{R}} \ll \underline{R} \quad .$$

These conditions define the domain of applicability of the perturbation method. From (2.4) and (2.5) we see that we may consider $\underline{R} \sim$ const. during times of the order of τ.

We now exhibit the consistency of these restrictive conditions by showing that they do indeed lead to the type of zero-order motion we wish to obtain.

For this purpose, we start with the Lagrangian

(2.6) $$L = \frac{m}{2} (\underline{\dot{r}} + \underline{\dot{R}})^2 + \frac{\varepsilon}{c} (\underline{\dot{r}} + \underline{\dot{R}}) \cdot \underline{A}(\underline{r} + \underline{R}) \quad .$$

We have seen that $\underline{R} \sim$ const. during times of the order of τ. Therefore, if we limit ourselves (in Hamilton's principle) to variations of the trajectory that are different from zero only during times of the order of τ, we may replace (2.6) by

(2.7) $$L \sim \frac{m}{2} \underline{\dot{r}}^2 + \frac{\varepsilon}{c} \underline{\dot{r}} \cdot \underline{A}(\underline{r} + \underline{R})$$

where only $\underline{r}$ is varied, and, after the variation, we use (2.2).

Thus, using (2.7) in Hamilton's principle and varying $\underline{r}$ only, we find:

(2.8) $$\underline{\ddot{r}} = \frac{\varepsilon}{mc} \underline{\dot{r}} \times \underline{H}(\underline{r} + R)$$

which, according to (2.2), is equivalent to

(2.9) $$\underline{\ddot{r}} = \frac{\varepsilon}{mc} \underline{\dot{r}} \times \underline{H}(\underline{R})$$
$$= \underline{\Omega}(\underline{R}) \times \underline{\dot{r}}$$

where

$$\underline{\Omega}(\underline{R}) = - \frac{\varepsilon \underline{H}(\underline{R})}{mc} \quad .$$

These equations are solved by:

$$\underline{\dot{r}} = \underline{\Omega}(\underline{R}) \times \underline{r} \quad . \tag{2.10}$$

That is, the particle revolves in a circle, of radius r, about the field lines. The angular velocity is $\underline{\Omega}$.

Thus, the restrictive conditions we have postulated do lead to a zero-order motion that takes place in a homogeneous field. We note, from (2.10), that the zero-order motion is entirely transverse to the magnetic field. In zero order, then, the motion along the magnetic field is undetermined. We shall show below that the first-order equations of motion fix the motion along the magnetic field lines completely.

We return to the Lagrangian (2.6). This time, we wish to vary $\underline{R}$. According to (2.5) only variations of $\underline{R}$ that involve many periods, τ, of the zero-order motion will be noticeable. That is, during a variation of $\underline{R}$, we may replace quantities involving $\underline{r}$ or $\underline{\dot{r}}$ by their average values. We therefore proceed to the calculation of the required averages.

First, we note that the term $m\underline{\dot{r}}^2/2$ in the Lagrangian may be ignored, since only terms involving $\underline{R}$ and $\underline{\dot{R}}$ will be varied. Next, we note that (2.1) implies a similar restriction on the vector potential:

$$\underline{r} \cdot \nabla_{\underline{R}} \, \underline{A}(\underline{R}) \ll \underline{A}(\underline{R}) \; .$$

Finally, since the zero-order motion given by (2.10) is such that the particle executes a circle about the field lines, we have:

$$\overline{\underline{r}} = 0 = \overline{\underline{\dot{r}}} \; .$$

From these remarks, the effective Lagrangian for variations of $\underline{R}$ is

$$\overline{L} \sim \frac{m}{2} \underline{\dot{R}}^2 + \frac{\varepsilon}{c} \underline{\dot{R}} \cdot \underline{A}(\underline{R}) + \frac{\varepsilon}{c} \overline{\underline{r} \cdot \nabla_{\underline{R}} \, \underline{A}(\underline{R}) \cdot \underline{\dot{r}}} \tag{2.11}$$

where the bar indicates an average over the zero-order motion.

3. The 'equivalent dipole'

We shall now show that

$$\frac{\varepsilon}{c}\,\overline{\underline{r} \circ \nabla_{\underline{R}}\, \underline{A}(\underline{R}) \cdot \dot{\underline{r}}} = \underline{\mu} \cdot \underline{H}(\underline{R}) \tag{3.1}$$

where

$$\underline{\mu} = \frac{\varepsilon}{2c}\, \overline{r^2}\, \underline{\Omega}(\underline{R})$$

is the magnetic moment of the 'equivalent dipole'.

For the left side of (3.1), we have, using (2.10),

$$\frac{\varepsilon}{c}\,\overline{\dot{\underline{r}}\,\underline{r}} : \nabla_{\underline{R}}\, \underline{A}(\underline{R}) = \frac{\varepsilon}{c}\, \underline{\Omega} \times \overline{\underline{r}\,\underline{r}} : \nabla_{\underline{R}}\, \underline{A}(\underline{R}) \; .$$

We therefore have to compute the dyadic $\overline{\underline{r}\,\underline{r}}$, where $\underline{r}$ is a vector in the plane perpendicular to the field $\underline{H}$, and the average is taken over a circle in this plane. We shall show that

$$\overline{\underline{r}\,\underline{r}} = \frac{\overline{r^2}}{2}\, I_t \tag{3.2}$$

where I_t is the unit dyadic in the plane perpendicular to $\underline{H}$.

To prove (3.2), we note that $\overline{\underline{r}\,\underline{r}}$ is, by definition, the two-dimensional dyadic $\underline{r}\,\underline{r}$ averaged over all directions in the plane. The only dyadic that is independent of direction in the plane is I_t; therefore $\overline{\underline{r}\,\underline{r}}$ is proportional to I_t. The constant of proportionality is evaluated by requiring the spur of both sides of (3.2) to be the same. This proves (3.2).

On the other hand, if a current I threads a closed contour C, the magnetic moment, $\underline{\mu}$, is given by:

$$\underline{\mu} = \frac{I}{2} \int_c \underline{r} \times d\underline{r} \ .$$

In our problem,

$$I = \frac{\varepsilon}{c\tau} \ .$$

Hence, using (2.10),

(3.3) $$\underline{\mu} = \frac{\varepsilon}{2c} \cdot \frac{1}{\tau} \int_0^{\tau} \underline{r} \times \dot{\underline{r}} \, dt$$

$$= \frac{\varepsilon}{2c} \cdot \overline{r^2} \, \underline{\Omega}(\underline{R}) \ .$$

Assembling these results, we see that the left side of (3.1) leads to the right side.

Returning to (2.11), the effective Lagrangian for variations of $\underline{R}$ becomes:

(3.4) $$\overline{L} \sim \frac{m}{2} \dot{\underline{R}}^2 + \frac{\varepsilon}{c} \dot{\underline{R}} \cdot \underline{A}(\underline{R}) + \underline{\mu}(\underline{R}) \circ \underline{H}(\underline{R})$$

$$= \frac{m}{2} \dot{\underline{R}}^2 + \frac{\varepsilon}{c} \dot{\underline{R}} \cdot \underline{A}(\underline{R}) - \mu(\underline{R}) \, H(\underline{R}) \ .$$

In the second form, where the scalars μ and H appear, we have used the fact that $\underline{\mu}$ and $\underline{H}$ are anti-parallel.

4. The constancy of $\mu(\underline{R})$

According to (3.3), the magnetic moment of the 'equivalent dipole' is a function of $\underline{R}$. We would therefore expect $\mu(\underline{R})$ to give a contribution to (3.4) when $\underline{R}$ is varied. In actuality, however, this is not the case: μ may be treated as a constant in (3.4) if the variations of $\underline{R}$ are sufficiently slow. We shall now carry out the reasoning that leads to this conclusion.

We first consider a homogeneous field along the z-axis, and we introduce the cylindrical coordinates (ρ, θ, z). The z-component of (canonical) angular momentum, P_θ, is constant. That is,

$$P_\theta = m\rho^2\dot{\theta} + \frac{\varepsilon}{c} A_\theta \rho \tag{4.1}$$

$$= m\rho^2\dot{\theta} + \frac{\varepsilon}{2c} H\rho^2 = \text{const.}$$

But, from (2.9),

$$\dot{\theta} = -\Omega = -\frac{\varepsilon H}{mc} \quad .$$

Therefore,

$$P_\theta = -\frac{m\rho^2}{2}\Omega = -\frac{mc}{\varepsilon}\mu = \text{const.} \tag{4.2}$$

The action variable associated with the coordinate θ is

$$J_\theta = \int_0^{2\pi} P_\theta \, d\theta = 2\pi P_\theta \tag{4.3}$$

$$= -\frac{mc}{\varepsilon} 2\pi\mu \quad .$$

For a homogeneous field, therefore, the magnetic moment of the equivalent dipole is a constant of the motion and is an action variable. It follows from the adiabatic theorem that it is also an adiabatic invariant, that is, the magnetic moment is unaffected by all changes that are 'adiabatically slow' compared with the period of the motion corresponding to an homogeneous field.

According to (2.5), we have limited ourselves to motions of $\underline{R}$ that are negligibly small in a period of the zero-order motion. Thus, the possible variations of R in (3.4) will satisfy the requirements of the adiabatic theorem. For our applications, therefore, μ may be considered a constant when R is varied in (3.4).

For completeness, we note that μ is related to the flux encircled by the particle, Φ, by:

$$\mu = \frac{m}{2\pi} \left(\frac{\varepsilon}{mc}\right)^2 \Phi \, . \tag{4.5}$$

Thus, the flux encircled by the particle during a period of the zero-order motion is also an adiabatic invariant.

As a final remark in this section, we note that $W_\perp$, the mean energy of motion transverse to the field, is given by:

$$W_\perp = - \underline{\mu} \cdot \underline{H} = \frac{m}{2} \overline{r^2} \, \Omega^2 = \frac{m}{2} \overline{\dot{r}^2} \, . \tag{4.6}$$

On the other hand, the total energy, E, is given by:

$$E = \frac{m}{2} \overline{(\dot{\underline{r}} + \dot{\underline{R}})^2} = \frac{m}{2} \dot{R}^2 + \frac{m}{2} \overline{\dot{r}^2} = W_{||} + W_\perp \, . \tag{4.7}$$

That is, motion along the lines of force is associated with $\underline{R}$. Although this already implies that the 'equivalent dipole' moves so as always to remain on the same line of force, we shall prove this more directly in the next section.

5. The orbit

We introduce a system of spherical coordinates, (R, Θ, ϕ) and we suppose the vector potential to have only the component A_ϕ different from zero. Since the vector potential is chosen divergenceless, it is independent of ϕ, and we are considering a situation where the field is unaffected by a rotation about the polar axis. It follows that P_ϕ, the (canonical) angular momentum about the polar axis, is a constant of the motion.

Introducing spherical coordinates into (3.4), we find:

$$P_\phi = \frac{\partial \overline{L}}{\partial \dot{\phi}} = m \, R \sin\Theta \left\{ R \sin\Theta \, \dot{\phi} + \frac{e}{mc} A_\phi \right\} = \text{const.} \tag{5.1}$$

Since

$$H \sim \frac{A}{R}$$

the second term in (5.1) is of the order

$$\frac{eH}{mc}\ R = \Omega\, r\left(\frac{R}{r}\right) = \dot{r}\left(\frac{R}{r}\right) .$$

The first term in (5.1), however, is of the order of $\dot{R}$. Therefore, according to (2.2) and (2.4), the second term is very much larger than the first and (5.1) becomes:

$$P_\phi \sim \frac{e}{c}\ A_\phi\ R\ \sin\theta = \text{const.} \tag{5.2}$$

The 'equivalent dipole' therefore moves so as to satisfy (5.2).

To see what this means geometrically, we observe that the field has the components

$$H_R = \frac{1}{R^2 \sin\theta}\ \frac{\partial}{\partial\theta}\ (A_\phi\ R\ \sin\theta)$$

$$H_\theta = -\frac{1}{R\sin\theta}\ \frac{\partial}{\partial R}\ (A_\phi\ R\ \sin\theta) \tag{5.3}$$

$$H_\phi = 0 .$$

If we make a displacement along a line of force, the components δR and $\delta\theta$ of the displacement are related to the field by:

$$\frac{\delta R}{R\delta\theta} = \frac{H_R}{H_\theta} .$$

Substituting from (5.3) in this relation, we find:

$$\left\{\delta R\ \frac{\partial}{\partial r} + \delta\theta\ \frac{\partial}{\partial\theta}\right\}(A_\phi\ R\ \sin\theta) = 0 . \tag{5.4}$$

Thus, (5.2) holds if and only if the 'equivalent diple' always moves on the surface generated by rotating a line of force about the polar axis.

In spherical coordinates, (3.4) has the form:

$$\bar{L} = \frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + m R \sin\theta\, \dot{\phi}\left\{\frac{R\sin\theta}{2}\dot{\phi} + \frac{\varepsilon}{mc}A_\phi\right\} - \mu H \ . \tag{5.5}$$

This Lagrangian corresponds to a particle-charge ε, mass m, magnetic moment μ - moving in the magnetic field $\underline{H}$.

By the same reasoning that led from (5.1) to (5.2), (5.5) may be replaced by

$$\bar{L} = \frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{\varepsilon}{c}A_\phi R \sin\theta\, \dot{\phi} - \mu H \ . \tag{5.6}$$

The energy integral corresponding to (5.6) is:

$$E = \frac{m}{2}(\dot{R}^2 + R^2\dot{\theta}^2) + \mu H \ . \tag{5.7}$$

The actual orbit of the 'equivalent dipole' may be obtained as follows: From (5.6) we obtain an equation of motion for each of the three coordinates. The equation for ϕ integrates at once to give (5.2). As already discussed, this prescribes the surface on which the 'equivalent dipole' moves. The form of this surface, as given by (5.2), may be written

$$R = f(\theta). \tag{5.8}$$

Putting this into (5.7), results in an equation for $\dot{\theta}$. Finally, using this new equation to eliminate $\dot{\theta}$ and $\ddot{\theta}$ from the equations of motion for the R and θ co-ordinates, previously derived from (5.6), we obtain an equation for $\dot{\phi}$. Integrating these equations for $\dot{\theta}$ and $\dot{\phi}$, and combining with (5.8), specifies the orbit completely.

References

[1] Alfven, H. - Cosmical electrodynamics; Oxford (1950).

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